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How long is a game of snakes and ladders?

S.C. ALTHOEN, L. KING and K. SCHILLING

"There is really only one Game, the Game in which each of us is a player acting out his role. The Game is Leela, the universal play of cosmic energy." Thus begins Harish Johari in *Leela: the game of self-knowledge*, a serious commentary and religious interpretation of the Hindu board game *Leela*. Leela is essentially the game *Snakes and Ladders*, which in the U.S. is the popular children's game *Chutes and Ladders*, first marketed as "the improved new version of snakes and ladders, England's famous indoor sport," for 50¢ in 1943 by the Milton Bradley Company of Springfield, Massachusetts. This game is really a 101 state absorbing Markov chain, which is amenable to mathematical as well as moral analysis. In what follows we investigate only the mathematical side of this diversion, specifically the expected playing time.

Description

The game we discuss is played on a ten by ten board with squares numbered 1 to 100. Starting off the board (in state 0), players in turn roll a six-faced die or use a spinner to determine how many squares to advance. Two or more tokens may occupy the same square simultaneously. The player to first reach square 100 *exactly* wins. If a move would take the player beyond the last square, the token is not moved. If a token lands on a square containing the bottom of a ladder it is immediately moved to the square at the top of the ladder. Similarly if the token lands on the square at the top of a snake, it is immediately moved to the square at the bottom of the snake. The game contains 10 snakes and 9 ladders with beginning and ending states given by the following ordered pairs:

Ladders:
(1,38) (4,14) (9,31) (21,42) (28,84) (36,44) (51,67) (71,91) (80,100)

Snakes:
(16,6) (47,26) (49,11), (56, 53) (62,19) (64,60) (87,24) (93,73)(95,75) (98,78)

In the American version the snakes are called chutes. The Hindu version is *Snakes and Arrows*. As a lesson in morality, Snakes and Ladders provides a list of good behaviours (bottom of ladders) and their rewards (tops of ladders). Similarly for snakes and bad behaviours. Thus, the ladder (4,14) represents "Study" leading to "Knowledge" and the snake (64,60) represents "Carelessness" leading to "Injury". The Milton

Bradley game merely has cartoon illustrations of behaviours at the beginning and ends of the chutes and ladders. There are other versions of the game. For example, *Leela* has 73 states (0 through 72) with 10 snakes and 10 arrows. State 68 (Cosmic Consciousness) is the winning state (there is a snake in state 72 to get you back to earth (state 51)) and there are special rules regarding the roll of a six. In the U.S. there is even a 7 by 7 travel version with 50 states, four snakes (called slides) and four ladders (called steps).

Though Snakes and Ladders is intended to be played by two or more players, we will consider a solitaire version, since opponents act independently as they move their tokens along the board. The question is, how long does it typically take a single player to finish a game by reaching square 100?

Simulation

It is easy to simulate any version of the game Snakes and Ladders using any programming language that has a random number generator. In the BASIC programming language, the function RND(X) is used to generate random numbers. But be careful – in most versions of BASIC each time you run a program, RND(X) generates the same "random" numbers. Even changing the "seed" X doesn't help – it is ignored. There are other dangers in using random number generators, but considering them would take us too far afield.

We simulated the Milton Bradley game using a version of BASIC called UBASIC86 (created by Yuji Kida of Kanazawa University) because its random number generator can produce distinct sequences. Here are the results of a simulation of 10 runs of 1000 games each:

Run Number	Average Length	Shortest Game	Longest Game
1	39.8	7	175
2	39.2	7	185
3	38.7	7	158
4	39.5	7	205
5	38.5	7	187
6	38.3	7	198
7	39.5	8	207
8	38.8	7	176
9	39.9	7	185
10	39.0	7	242

Overall average length: 39.1

It seems safe to say that the average length of a game of Snakes and Ladders is about 39 moves. It is not hard to see that the shortest possible game lasts only seven moves, and that there are several ways to obtain this minimum. (Rolls of 1, 2, 5, 6, 5, 5, 3 and 4, 6, 6, 2, 6, 4, 6 are two of them). Of course, there is no longest game.

Theory of Markov Chains

One can use the theory of Markov chains in a standard way to compute the exact expected (i.e. average) length of a game of Snakes and Ladders. Although the game board for Snakes and Ladders has 100 squares, it is most natural to model the game as a Markov chain with 82 states. This is because we first add a State 0, the player's state before his first roll puts him on the board. We then delete each state corresponding to the beginning of a snake or ladder. For example, a player is never actually in State 1, for if he lands on square 1 the ladder (1,38) immediately deposits him into State 38.

The transition matrix for a Markov chain with n states is an $n \times n$ matrix where the entry in row i and column j is the probability that a player in state i on one turn will be in state j on the next turn. Our Markov chain will be an 82×82 matrix with most rows containing 6 consecutive $^1/_6$'s, with two kinds of exceptions. First, the row corresponding to any state from which you can reach a snake or ladder must be changed to take into account the effect of that snake or ladder. Second, the last few rows must reflect the fact that a player must reach State 100 exactly; a die roll which would cause the player to go past State 100 results in no move. We leave it to the reader to construct the entire transition matrix for Snakes and Ladders. As a hint, here are the right ends of the last few rows labelled with the corresponding squares:

$$
\begin{array}{c|ccccccccccccccccccccc}
 & 75 & 76 & 77 & 78 & 79 & 81 & 82 & 83 & 84 & 85 & 86 & 88 & 89 & 90 & 91 & 92 & 94 & 96 & 97 & 99 & 100 \\
\hline
94 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
96 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
97 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} \\
99 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\
100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

The theory of Markov chains says that we may compute the expected (average) playing time for this game as follows:

Let M be the 81×81 matrix obtained by deleting the last row and column from the transition matrix. Let I be the 81×81 identity matrix. Let $(s_{ij}) = (I - M)^{-1}$. Then s_{0j} is the expected number of rolls the player will spend in square j. The expected number of rolls needed to complete the game is the sum of the entries in the top row of the matrix $(I - M)^{-1}$.

We performed this computation on a Zenith microcomputer, again using the language UBASIC86. In order to avoid roundoff error, we used exact computations; that is, instead of storing the decimal approximation of each number, we stored its exact numerator and denominator. (Of course, no irrational numbers arise in the computation of the inverse of a matrix with integer entries. We chose UBASIC86 because it allows very large integers.) The exact expected number of moves in the Milton Bradley version of Chutes and Ladders is

$$\frac{225837582538403273407117496273279920181931269186581786048583}{57574729981400392329505758746287861311130999406013041613400}$$

which is approximately 39.2. This is reasonably close to our simulations in above.

Sensitivity

What effect does adding a single snake or ladder have on the average length of a game? In this section we show how to answer this question.

Consider a general Snakes and Ladders-style game with n squares, played with an m-sided die. We will model the game with a slightly different Markov chain from the one we used in before.

First consider the game with no snakes or ladders. In this case, as above, our Markov Chain has $n+1$ states, and its $(n+1) \times (n+1)$ transition matrix has rows

$$(0, \ldots, 0, 1/m, \ldots, 1/m, 0, \ldots, 0)$$

with m consecutive $1/m$'s – except in the last few rows, when the 'must land exactly on square n' rule takes effect. If M is the matrix obtained by deleting the last row and column of this transition matrix, it is not hard to see that $(I - M)^{-1} = (s_{ij})$, where

$$s_{jj} = 1 \text{ for } 0 \leqslant j < n-m; \; s_{jj} = \frac{m}{n-j} \text{ for } n-m \leqslant j < n$$

$$s_{jk} = \frac{1}{m} \sum_{i=k-m}^{k-1} s_{ji} \quad \text{for } j < k \leqslant n-m$$

$$s_{ji} = 0 \text{ for } i < j$$

$$s_{jk} = \frac{1}{n-k} \sum_{i=k-m}^{k-1} s_{ji} \quad \text{for } n-m < k < n.$$

These values were obtained by Gaussian elimination.

We used UBASIC86 and these formulas to find the expected playing time for a 100-square game played with a six-sided die. (That is, Snakes and Ladders with neither snakes nor ladders!) It is almost exactly 33 moves. You may recall that the average length of a game of Snakes and Ladders was about 39 moves. Apparently the snakes lengthen the game more than the ladders shorten it.

Next, let's add snakes and/or ladders. For each snake or ladder, say from square i to square j, change row i of the transition matrix to

$$(0, ..., 0, 1, 0, ..., 0)$$

where the 1 is in column j. The matrix remains $(n + 1) \times (n + 1)$, and the matrix M obtained by removing its last row and column will be $n \times n$.

By changing instead of eliminating row i, we have modelled a game where one turn is spent at the top of a snake or bottom of a ladder, and the next turn is spent at its other end. Therefore, if we have snakes and ladders from states in the set $\mathfrak{S}$, the exact playing time of the game is

$$\sum_{l \notin \mathfrak{S}} s_{0l}$$

where $(s_{ij}) = (I - M)^{-1}$.

Now suppose that we fix a game with matrix M as in the preceding paragraph. Let $S = I - M$, and $(s_{kl}) = S^{-1}$. Suppose that we know all of the entries (s_{kl}). What is the effect of adding a new snake or ladder from state i to state j ?

To answer this question, let M' be the transition matrix (minus last row and column) of the game with the new snake or ladder, We need to compute $S'^{-1} = (s'_{kl})$ where $S' = I - M'$. Let $E = S' S^{-1}$, so $ES = S'$.

Since S and S' agree in all but the i^{th} row, E must be the identity matrix in all but the i^{th} row:

$$e_{kl} = \delta_{kl} \quad k \neq i,$$

for δ_{kl} the Kronecker delta; that is, 1 if $k = l$, 0 if $k \neq l$. The i^{th} row of E is the i^{th} row of S' times S^{-1}.

The i^{th} row of $S' = I - M'$ is

$$(0, ..., \overset{i}{1}, 0, ..., 0, \overset{j}{-1}, 0, ... , 0)$$

$$(0, ..., \overset{j}{-1}, 0, ..., 0, \overset{i}{1}, 0, ... , 0)$$

depending upon whether $i < j$ or $i > j$. If we multiply either of these rows by the l^{th} column of S^{-1}

$$\begin{bmatrix} s_{1l} \\ \cdot \\ \cdot \\ \cdot \\ s_{nl} \end{bmatrix}$$

we obtain $e_{il} = s_{il} - s_{jl}$.

Since $S'^{-1} = S^{-1}E^{-1}$, to obtain S'^{-1} from S^{-1} we need only calculate E^{-1}. The matrix E is the $n \times n$ identity matrix with its i^{th} row replaced by $e_{il} = s_{il} - s_{jl}$. Think of computing E^{-1} by Gauss-Jordan reduction. In each column except column i we would use the row operation $R_i - e_{il}R_l$. These operations reduce E to the identity except for the e_{ii} left on the diagonal.

To complete the reduction we divide row i by e_{ii} . Thus $E^{-1} = (f_{ij})$, where

$$f_{ik} = \delta_{kl} \quad \text{for } k \neq i$$

$$f_{il} = -\frac{e_{il}}{e_{ii}} = -\frac{s_{il} - s_{jl}}{s_{ii} - s_{ji}} \quad \text{for } l \neq i$$

$$f_{ii} = \frac{1}{e_{ii}} = \frac{1}{s_{ii} - s_{ji}} .$$

Note that $\det E = e_{ii} = s_{ii} - s_{ji} \neq 0$, since E is invertible. Finally, we compute $S'^{-1} = S^{-1}E^{-1}$. Let $S'^{-1} = (s'_{kl})$.Then by considering the two cases $l \neq i$ and $l = i$ we obtain the answer:

$$s'_{kl} = \begin{cases} s_{kl} + s_{ki}\left(\dfrac{s_{jl} - s_{il}}{s_{ii} - s_{ji}}\right) & \text{for } l \neq i \\ \dfrac{s_{ki}}{s_{ii} - s_{ji}} & \text{for } l = i . \end{cases}$$

This formula can be used to calculate quickly and exactly the effect of adding a snake or ladder on, say, Snakes and Ladders. To save space in what follows we have rounded the exact rational answers. In general, one expects that an additional ladder will shorten the game and an additional snake will lengthen it. For example, if we insert a ladder from state 46 to state 94 the expected length of the game drops from about 39.2 to about 29.8; if instead we insert a snake from state 83 to state 7, the game lengthens to about 45.8 moves. Surprisingly, this is not always the case. Adding a ladder to the original game from state 79 to state 81 lengthens the expected playing time by more than two moves to about 41.9, since it increases the chances of missing the important ladder from state 80 to 100. Similarly, adding a snake to the original game from state 29 to state 27 shortens the game by over a move to about 38.0, since it gives a second chance at the long ladder from 28 to 84.

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