1 Norms

1.1 Definition

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $\mathbf{dom} f = \mathbb{R}^n$ is called a *norm* if

- f is nonnegative: $f(x) \geq 0$ for all $x \in \mathbb{R}^n$
- f is definite: $f(x) = 0$ only if $x = 0$
- f is homogeneous: $f(tx) = |t|f(x)$, for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$
- f satisfies the triangle inequality: $f(x+y) \leq f(x)+f(y)$, for all $x, y \in \mathbb{R}^n$

1.2 Unit ball

The set of all vectors with norm less than or equal to one,

$$\mathcal{B} = \{x \in \mathbb{R}^n | \|x\| \leq 1\},$$

is called the *unit ball* of the norm $\|\cdot\|$.

The unit ball satisfies the following properties:

- $\mathcal{B}$ is symmetric about the origin, i.e., $x \in \mathcal{B}$ if and only if $-x \in \mathcal{B}$
- $\mathcal{B}$ is convex
- $\mathcal{B}$ is closed, bounded, and has nonempty interior

Conversely, if $C \subseteq \mathbb{R}^n$ is any set satisfying these three conditions, then it is the unit ball of a norm, which is given by

$$\|x\| = (\sup\{t \geq 0 | tx \in C\})^{-1}.$$

1.2.1 Exercise

Show triangle inequality is equivalent to convexity of norm ball.

1.3 Examples

- ℓ_1-norm, *sum-absolute-value*

$$\|x\|_1 = |x_1| + \cdots + |x_n|.$$

- ℓ_2-norm, *Euclidean*

$$\|x\|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

- ℓ_∞-norm, *Chebyshev*

$$\|x\|_\infty = \max\{|x_1|, \cdots, |x_n|\}.$$

- ℓ_p-norm, $p \geq 1$

$$\|x\|_p = (x_1^p + \cdots + x_n^p)^{1/p}.$$

Note that the above formula yields the ℓ_1-norm when $p = 1$ and the Euclidean norm when $p = 2$.

It is easy to show that for any $x \in \mathbb{R}^n$,

$$\lim_{p \to \infty} \|x\|_p = \max\{|x_1|, \cdots, |x_n|\},$$

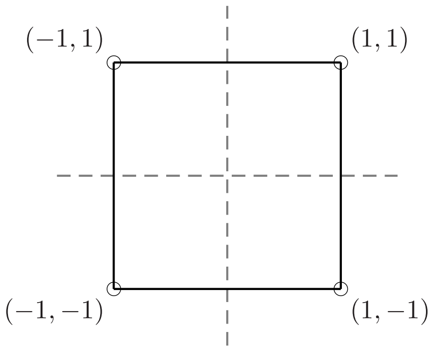
so the ℓ_p-norm also fits in this family, as a limit.

1.4 Illustration

$$x = (x_1, x_2)$$

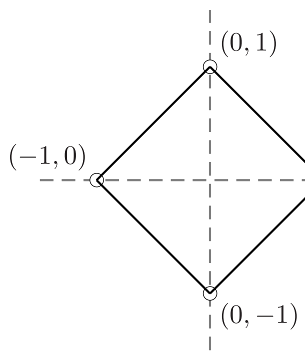
- $\|x\|_\infty = \max_i (x \cdot v_i)$

$$v_i = \{\pm(1,1), \pm(1,-1)\}$$



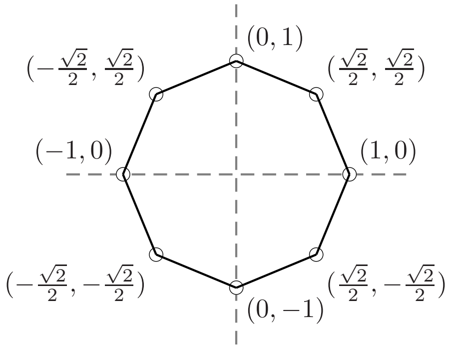
- $\|x\|_1 = \max_i (x \cdot v_i)$

$$v_i = \{(\pm 1, 0), (0, \pm 1)\}$$



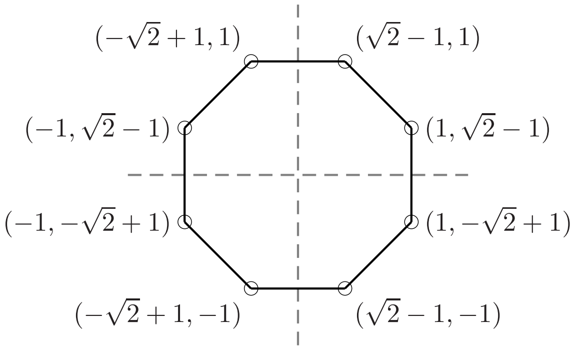
- $\|x\|_{\text{oct}} = \max_i(x \cdot v_i)$

$$v_i = \{\pm(0,1), \pm(1,0), \pm\frac{\sqrt{2}}{2}(1,1), \pm\frac{\sqrt{2}}{2}(1,-1)\}$$



- $\|x\|_{\text{oct}^*} = \max_i(x \cdot v_i)$

$$v_i = \{\pm(\sqrt{2}-1,1), \pm(1,\sqrt{2}-1), \pm(\sqrt{2}-1,-1), \pm(-1,\sqrt{2}-1)\}$$



1.4.1 Exercise

Show oct, oct*-norm are dual norms. One way to do this is by computing the formula directly. Another way to do this is by noting that the unit balls in the dual norms are polars of each other, and by using a formula which relates the linear inequalities which define a polyhedral set to the vertices of the polar. (See: Convexity Notes)

2 Dual norms

2.1 Definition

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The associated *dual norm*, denoted $\|\cdot\|_*$, is defined as :

$$\|z\|_* = \sup\{z^T x | \|x\| \leq 1\}.$$

2.2 Inequality

From the definition of dual norm we have th inequality $z^T x \leq \|x\| \|z\|_*$, which holds for all x and z. This inequality is tight, in the following sense: for any x there is a z for which the inequality holds with equality. (Similarly, for any z there is an x that gives equality.)

2.3 Property

- The dual of the dual norm is the original norm:

$$\|x\|_{**} = \|x\|$$

  for all x.

- The dual of the Euclidean norm is th Euclidean norm:

$$\sup\{z^T x | \|x\|_2 \leq 1\} = \|z\|_2.$$

- The dual of the ℓ_∞-norm is the ℓ_1-norm:

$$\sup\{z^T x | \|x\|_\infty \leq 1\} = \sum_{i=1}^{n} |z_i| = \|z\|_1,$$

  and the dual of the ℓ_1-norm is the ℓ_∞-norm.

- More generally, the dual of the ℓ_p-norm is the ℓ_q-norm, where q satisfies

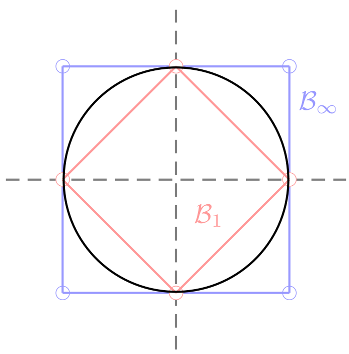
$$\frac{1}{p} + \frac{1}{q} = 1, \text{ i.e., } q = \frac{p}{p-1}.$$

2.3.1 Exercise

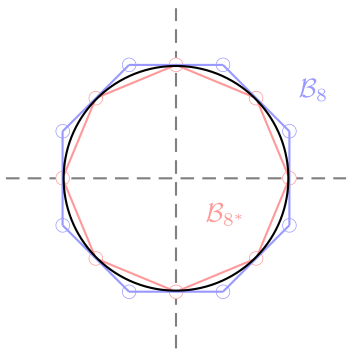
- Prove the dual of the dual norm is the original norm.
- Use Hölder's inequality to prove that $x \cdot y \leq \|x\|_p \|y\|_q$.

2.4 Illustration

- The dual of the ℓ_∞-norm is the ℓ_1-norm:



- $\mathcal{B}_8$ and $\mathcal{B}_{8^*}$:



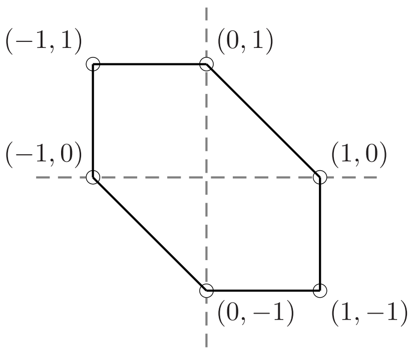
3 Exercise

1. Give the formula for the dual norm.
2. Give an example of a norm which is its own dual.
3. Consider the norm

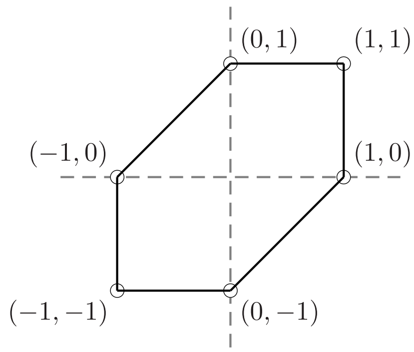
$$\|(x_1, x_2)\| = \max(|x_1|, |x_2|, |x_1 + x_2|).$$

   (a) Sketch the unit ball $\mathcal{B}$ in the norm. Indicate the vertices.
   (b) Find the dual of the norm.
   (c) Sketch the unit ball in the dual norm, and indicate the vertices.

Unit ball $\mathcal{B}$



Dual norm unit ball



4 References

S. Boyd, L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004. [http://www.stanford.edu/ boyd/cvxbook/]